Criteria of unextendability of the images of real analytic maps and their application for constant mean curvature one surfaces in S^3_1

Subtitle: **Analytic extensions of Catenoids in $\mathbf{R}^3_1$ and S^3_1**

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(To explain unextendability of the images of real analytic maps, we must give real analytic maps whose images have non-trivial analytic extensions!)

This is a joint work [1] with

S, Fujimori, M. Kokubu, Y. Kawakakmi, W. Rossman, K. Yamada and S. D. Yang.

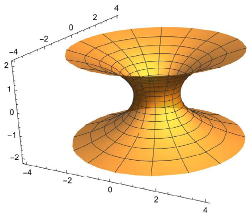


図 1. The catenoid in $\mathbf{R}^3$, $f_0 = (\cosh r \cos\theta, \cosh r \sin\theta, r)$.

References

[1] S. Fujimori, M. Kokubu, Y. Kawakakmi, W. Rossman, K. Yamada and S. D. Yang, *Analytic extensions of constant mean curvature one geometric catenoids in de Sitter 3-space*, preprint (arXiv:2011.06757).

[2] O. Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space L^3*, Tokyo J. Math., **6** (1983), 297–309.

[3] S. Fujimori, *Spacelike CMC 1 surfaces with elliptic ends in de Sitter 3-space*, Hokkaido Math. J., **35** (2006), 289–320.

[4] S.-D. Yang, *Björling formula for mean curvature one surfaces in hyperbolic three-space and in de Sitter three space*, Bull. Korean Math. Soc. **54** (2017), 159–175.

Catenoids as Maximal surfaces in $\mathbf{R}^3_1$

The image of Euclidean catenoid is invariant under the rotation with respect to the z-axis.

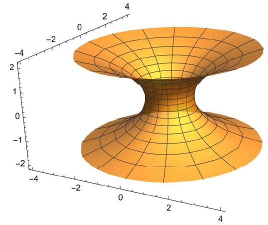


図 2. The catenoid in $\mathbf{R}^3$, $f_0 = (\cosh r \cos\theta, \cosh r \sin\theta, r)$.

$$\mathbf{R}^3_1 := (\mathbf{R}^3, (++-))$$

Maximal surfaces := Space-like zero-mean-curvature surfaces in $\mathbf{R}^3_1$.

$\mathbf{R}^3_1$-Catenoids := $\begin{pmatrix}\text{Maximal surfaces which are invariant under the one-parameter family} \\ \text{of isometries in } \mathbf{R}^3_1 \text{ with a common fixed point.}\end{pmatrix}$

Three types of 1-parameter groups in $\mathrm{SO}_+(2,1)$

The connected isometry group fixing the origin in $\mathbf{R}^3_1$:

$$\mathrm{SO}_+(2,1) := \{A \in \mathrm{GL}(3,\mathbf{R})\,;\, A^T E_3 A = E_3,\ \det(A) = 1,\ a_{33} < 0\},$$

$$\text{where } E_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Fact 1. *There are three types of 1-parameter groups in* $\mathrm{SO}_+(2,1)$*:*

$$E_t := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H_t := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}, \quad P_t := \begin{pmatrix} 1 & -t & t \\ t & (2-t^2)/2 & t^2/2 \\ t & -t^2/2 & (t^2+2)/2 \end{pmatrix},$$

where $t \in \mathbf{R}$.

- $\{E_t\}$ is said to be **elliptic** or of *Type* T,
- $\{H_t\}$ is said to be **hyperbolic** or of *Type* S,
- $\{P_t\}$ is said to be **parabolic** or of *Type* L.

In fact, E_t fixes $(0,0,1)$, H_t fixes $(1,0,0)$ and P_t fixes $(0,1,1)$.

Three types of $\mathbf{R}^3_1$-catenoids

There are 3 types of $\mathbf{R}^3_1$-catenoids corresponding to the 3 types of 1-paremeter groups in $SO_+(2,1)$.

(1) The elliptic catenoid, which is a C^ω-map

$$f_E(u,v) := (u\cos v, u\sin v, \sinh u) \qquad (u,v) \in \mathbf{R} \times \mathbf{R}/2\pi\mathbf{Z},$$

(2) The parabolic catenoid, which is a C^ω-map

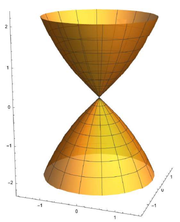
$$f_P(u,v) := \left(v + \frac{v^3}{3} - u^2 v, 2uv, v - \frac{v^3}{3} + u^2 v\right) \qquad (u,v) \in \mathbf{R}^2,$$

(3) The hyperbolic catenoid, which is a C^ω-map

$$f_H(u,v) := (v, \sinh u \sin v, \cosh u \sin v), \qquad (u,v) \in \mathbf{R}^2.$$

All of them are real analytic maps and are in Osamu Kobayashi [2].

Elliptic Catenoid



⊠ 3. The elliptic catenoid $f_E = (u\cos v, u\sin v, \sinh u)$ in $\mathbf{R}^3_1$

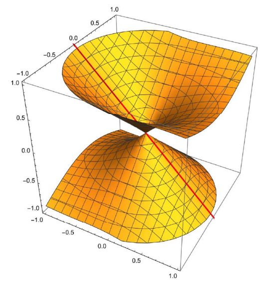
The image of f_E is rotationally symmetric with respect to the time-like axis, and has a cone-like singular point. The image of f_E coincides with the analytic set

$$\mathcal{E} = \{(x,y,t) \in \mathbf{R}^3_1\,;\, x^2 + y^2 = (\sinh^{-1} t)^2\}.$$

Thus,

Proposition 2. *The image of f_E has no analytic extension.*

Parabolic Catenoid



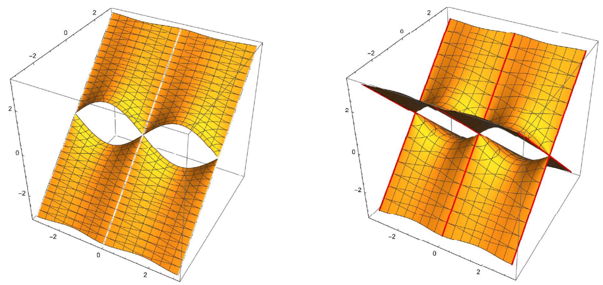
⊠ 4. The parabolic catenoid $f_P := \left(v + \frac{v^3}{3} - u^2 v, 2uv, v - \frac{v^3}{3} + u^2 v\right)$.

The image of f_P is invariant under the parabolic subgroup fixing $(-1,0,1)$, and $\mathcal{P} = \mathrm{Im}(f_P) \cup \mathcal{L}$ holds, where

$$\mathcal{P} := \left\{(x,y,t) \in \mathbf{R}^3_1\,;\, 12(x^2 - t^2) = (x+t)^4 - 12y^2\right\}, \quad \mathcal{L} := \left\{(0,t,-t)\,;\, t \in \mathbf{R}\right\}.$$

Proposition 3. *The set $\mathcal{P}$ has a cone-like singular point, and is an analytic extension of $\mathrm{Im}(f_P)$. Moreover, $\mathcal{P}$ has no further analytic extension.*

Hyperbolic Catenoid



⊠ 5. The hyperbolic catenoid $f_H := (v, \sinh u \sin v, \cosh u \sin v)$

The image of f_H is invariant under the hyperbolic subgroup fixing $(1,0,0)$. Moroever $\mathcal{H} = \overline{\mathrm{Im}(f_H) \cup T(\mathrm{Im}(f_H))}$ holds, where

$$T : (x,y,t) \mapsto (-x, y, -t), \quad \mathcal{H} := \left\{(x,y,t) \in \mathbf{R}^3_1\,;\, \sin^2 x + y^2 - t^2 = 0\right\}.$$

Proposition 4. *$\mathcal{H}$ has countably many cone-like singular points, and is an analytic extension of $\mathrm{Im}(f_H)$. Moreover, $\mathcal{H}$ has no further analytic extension.*

S^3_1-Catenoids

Let $\mathbf{R}^4_1$ be the Lorentz-Minkowski 4-space with the metric $\langle\ ,\ \rangle$ of signature $(-+++)$. The *de Sitter 3-space* is the subset

$$S^3_1 = \left\{ X := (x_0, x_1, x_2, x_3) \in \mathbf{R}^4_1 \,;\, -(x_0)^2 + \sum_{j=1}^{3} (x_j)^2 = 1 \right\},$$

with the metric induced from $\mathbf{R}^4_1$, is a simply-connected Lorentzian 3-manifold with constant sectional curvature 1.

CMC-1 surfaces := Space-like mean-curvature 1 surfaces in S^3_1.

S^3_1-Catenoids := (CMC-1 surfaces which are invariant under a 1-parameter family of isometries in S^3_1 with a common fixed point.)

S^3_1-Catenoids

- $\exists$ three types T, S and L of 1-parameter group of $SO^+(3,1)$.
- A CMC-1 surface is constructed from a holomorphic data (g, ω) (cf. Fujimori [3]). The meromorphic function g is called *the secondary Gauss map*, which may not be single-valued. The monodromy matrix of g at an end belongs to $\mathrm{SO}^+(3,1)$, which is elliptic (T), hyperbolic (S) or parabolic (L).

There are **8-types** of S^3_1-catenoids (Seong-Deog Yang [4]):

(1) Each of h_{TE}, h_{TH} and h_{TP} are invariant under a subgroup of type T. The secondary Gauss maps g of them have elliptic (type T), hyperbolic (type S) or parabolic (type P) monodromy, respectively.

(2) Each of h_{SE}, h_{SH} and h_{SP} are invariant under a subgroup of type S. The secondary Gauss maps g of them have elliptic (type T), hyperbolic (type S) or parabolic (type P) monodromy, respectively.

(3) Each of h_{LE} and h_{LH} are invariant under a subgroup of type L.

Theorem 5. *The images of catenoids of h_{SE}, h_{SH} and h_{SP} have no analytic extensions, and the remaining five catenoids admit analytic extensions.*
(We pick up h_{TE}, h_{TH} and h_{SE}, here!)

Catenoids of type TE

For each $\mu \in \mathbf{R}_+ \setminus \{1\}$, we set

$$h^\mu_{\mathrm{TE}} \colon \mathbf{R} \times \mathbf{R}/2\pi\mathbf{Z} \ni (s,\theta) \mapsto (x_0(s), x_1(s,\theta), x_2(s,\theta), x_3(s)) \in S^3_1,$$

which gives a S^3_1-catenoid of type TE, where

$$h^\mu_{\mathrm{TE}}(s,\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Gamma^\mu_{\mathrm{TE}}(s), \quad \Gamma^\mu_{\mathrm{TE}}(s) := \begin{pmatrix} x_0(s) \\ \frac{1-\mu^2}{2\mu} \sinh \mu s \\ 0 \\ x_3(s) \end{pmatrix},$$

and

$$x_0(s) := \sinh s \cosh \mu s - \frac{(\mu^2+1)\cosh s \sinh \mu s}{2\mu},$$

$$x_3(s) := \cosh s \cosh \mu s - \frac{(\mu^2+1)\sinh s \sinh \mu s}{2\mu}.$$

The catenoid h^μ_{TE} is a C^ω-map, which has a cone-like singular point. h^μ_{TE} is a proper map, and its image has no analytic extension.

Figures of the image of h^μ_{TE}

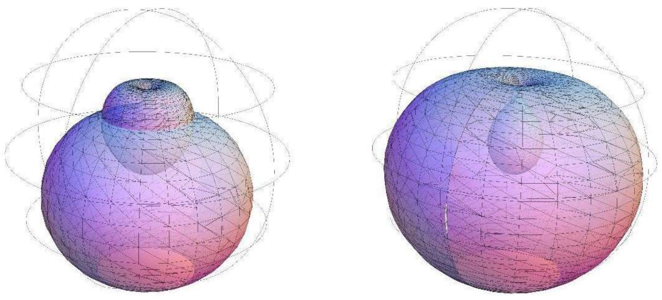


図 6. S^3_1-catenoids of type TE with $0 < \mu < 1$ (left) and $\mu > 1$ (right)

Catenoids of type TH

For each $\nu \in \mathbf{R}_+ \setminus \{1\}$, we set

$$h^\nu_{\rm TH}\colon \mathbf{R}\times\mathbf{R}/2\pi\mathbf{Z} \ni (s,\theta) \mapsto (x_0(s), x_1(s,\theta), x_2(s,\theta), x_3(s)) \in S^3_1,$$

which gives a S^3_1-catenoid of type TH, where

$$h^\nu_{\rm TH}(s,\theta) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Gamma^\nu_{\rm TH}(s), \quad \Gamma^\nu_{\rm TH}(s) := \begin{pmatrix} x_0(s) \\ \frac{\nu^2+1}{2\nu}\cos s \\ 0 \\ x_3(s) \end{pmatrix},$$

and

$$x_0(s) := \sin s \sinh(s/\nu) - \frac{(\nu^2-1)\cos s \cosh(s/\nu)}{2\nu},$$

$$x_3(s) := \sin s \cosh(s/\nu) - \frac{(\nu^2-1)\cos s \sinh(s/\nu)}{2\nu}.$$

The catenoid $h^\nu_{\rm TH}$ is a C^ω-map, ans is not a proper map, but has no analytic extension.

Properties of TH-catenoid $h^\nu_{\rm TH}$

We set

$$\mathcal{X}^\nu_{\rm TH} := \left\{ (\xi,\eta,s) \in \mathbf{R}^2 \times \mathbf{R}/2\pi\mathbf{Z}\,;\, \xi^2+\eta^2 = \frac{(1+\nu^2)^2\cos^2 s}{4\nu^2} \right\}$$

and consider an embedding

$$\Psi : \mathbf{R}^3 \ni (\xi,\eta,s) \mapsto (x_0(s), \xi, \eta, x_3(s)) \in S^3_1.$$

Then

$$h^\nu_{\rm TH}(s,\theta) := \Psi(x_1(s,\theta), x_2(s,\theta), s)$$

holds, and the map $h^\nu_{\rm TH}$ has countably many cone-like singular points.

When $\nu = 1$, $h^\nu_{\rm TH}$ has the following simple expression

$$h^1_{\rm TH}(s,\theta) = (\sin s \sinh s,\ \cos\theta\cos s,\ \sin\theta\cos s,\ \sin s\cosh s),$$

and

$$h^1_{\rm TH}(k\pi,\theta) = (0, (-1)^k\cos\theta, (-1)^k\sin\theta, 0)$$

for each $k \in \mathbf{Z}$, that is, $f^\nu_{\rm TH}$ $(\nu=1)$ takes the same value infinitely many times.

Catenoids of type SE

For each $\mu \in \mathbf{R}_+ \setminus \{1\}$, we can consider a S^3_1-catenoid of type SE.

$$h^\mu_{\rm SE}\colon \mathbf{R}^2 \ni (s,\theta) \mapsto (x_0(s,\theta), x_1(\theta), x_2(\theta), x_3(s,\theta)) \in S^3_1$$

given by

$$h^\mu_{\rm SE}(s,\theta) = \begin{pmatrix} \cosh s & 0 & 0 & \sinh s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh s & 0 & 0 & \cosh s \end{pmatrix} \Gamma^\mu_{\rm SE}(\theta), \quad \Gamma^\mu_{\rm SE}(\theta) := \begin{pmatrix} -\frac{\mu^2-1}{2\mu}\cos\mu\theta \\ x_1(\theta) \\ x_2(\theta) \\ 0 \end{pmatrix},$$

where

$$x_1(\theta) := -\frac{(\mu^2+1)\cos\theta\cos\mu\theta}{2\mu} - \sin\theta\sin\mu\theta,$$

$$x_2(\theta) := -\frac{(\mu^2+1)\sin\theta\cos\mu\theta}{2\mu} + \cos\theta\sin\mu\theta.$$

This map is obtained by the Weierstrass type representation as well as $h^\mu_{\rm TE}$ and $h^\nu_{\rm TH}$ given by Fujimori [3]. $h^\mu_{\rm SE}$ has an analytic extension.

Properties of SE-catenoid $f^\mu_{\rm SE}$

We set

$$\mathcal{X}^\mu_{\rm SE} := \left\{ (x,y,\theta) \in \mathbf{R}^3\,;\, x^2 - y^2 = \left(\frac{(\mu^2-1)\cos\mu\theta}{2\mu} \right)^2 \right\}$$

and consider the embedding

$$\Psi : \mathbf{R}^3 \ni (x,y,\theta) \mapsto \left(\frac{y}{2}, x_1(\theta), x_2(\theta), \frac{x}{2} \right) \in S^3_1.$$

Then

$$h^\mu_{\rm SE}(s,\theta) = \Psi\left(\frac{\mu^2-1}{2}\sinh s, \frac{\mu^2-1}{2}\cosh s, \theta \right)$$

holds and

$$\Psi(\mathcal{X}^\mu_{\rm SE}) = {\rm Im}(h^\mu_{\rm SE}) \cup T({\rm Im}(h^\mu_{\rm SE})) \cup \mathcal{L}$$

gives the analytic extension of ${\rm Im}(h^\mu_{\rm SE})$, where

$$\mathcal{L} := \bigcup_{k\in\mathbf{Z}} \mathcal{L}_k, \quad \mathcal{L}_k := \left\{ \left(u, \gamma^\mu_{\rm SE}\left(\frac{\pi}{2\mu}(2k+1) \right), \pm u \right) \in S^3_1\,;\, u \in \mathbf{R} \right\},$$

and

$$T(x_1,x_2,x_3,t) = (-x_1,x_2,x_3,-t).$$

The sufficiency of the analytic extensions of S^3_1-catenoids.

N^n; a real analytic n-manifold.

Definition 6. A subset S of N^n is said to be *analytically complete* if it satisfies the following property:

Any real analytic map $\Gamma : [0,1] \to N^n$ satisfying $\Gamma([0,\varepsilon)) \subset S$ for some $\varepsilon \in (0,1)$ satisfies $\Gamma([0,1]) \subset S$.

For example:

- Any linear subspace of $\mathbf{R}^n$ is an analytically complete.
- Any subinterval $I \subsetneq \mathbf{R}$ is not an analytically complete subset of $\mathbf{R}$. In fact, if we set $\Gamma(t) := t$, then $\Gamma(I) \subset I$ but $\Gamma(\mathbf{R}) \not\subset I$.

Proposition 7. *If S^3_1-catenoids or their analytic extensions are analytically complete, we can say "they have no further analytic extensions".*

Arc-properness

N^n; a real analytic n-manifold.
X; a metrizable topological space.
Fix $r \in \mathbf{N} \cup \{\infty, \omega\}$. In FKKRUYY-paper [4], we defined the following:

Definition 8. Let $f : X \to N^n$ be a continuous map. A continuous curve $\sigma : [0,1) \to X$ is said to be *(C^r, f)-extendable* if $f \circ \sigma(t)$ can be extended to a C^r-map defined on $[0,1]$, that is, there exist an open interval J containing $[0,1]$ and a C^r-map Γ such that $\Gamma|_{[0,1)} = f \circ \sigma$.

Definition 9. Let $f : X \to N^n$ be a continuous map. Then f is called *C^r-arc-proper* if for each (C^r, f)-extendable continuous curve $\sigma : [0,1) \to X$, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ on $[0,1)$ converging to 1 such that $\lim_{k\to\infty} \sigma(t_k) \in X$ exists.

Proposition 10. *Properness implies C^r-arc-properness. If $r < s$, then C^r-arc-properness implies C^s-arc-properness.*

So, C^ω-arc-properness is the weakest condition.

A criterion for analytic completeness

Theorem 11 (FKKRUYY [1]). *Let $f : M^m \to N^n$ be a C^ω-map which admits only cone-like singular points. If f is a C^ω-arc proper map, then $f(M^m)$ is analytical complete, that is, $f(M^m)$ has no analytic extensions.*

Example 12. We consider the image $f(\mathbf{R})$ of the C^ω-immersion

(1) $$f : \mathbf{R} \ni t \mapsto (t, \sqrt{2}t) \in T^2 := \mathbf{R}^2/\mathbf{Z}^2.$$

Although f is **not a proper map**, f is C^0-arc-proper:

$\Gamma : [0,1] \to T^2$, a C^0-map satisfying $\Gamma([0,1)) \subset f(\mathbf{R})$.
$\gamma : [0,1) \to \mathbf{R}$, a C^0-map satisfying $f \circ \gamma = \Gamma$ on $[0,1)$.
$p_1 : T^2 \ni (x,y) \mapsto x \in \mathbf{R}$, the first projection. Then $p_1 \circ \Gamma(t) = \gamma(t)$ on $[0,1)$, and so

$$\lim_{t\to 1-0} \gamma(t) = \lim_{t\to 1-0} p_1 \circ \Gamma(t) = p_1 \circ \pi \circ \tilde{\Gamma}(1).$$

Thus, f is C^0-arc-proper, and is analytically complete.

Example 2

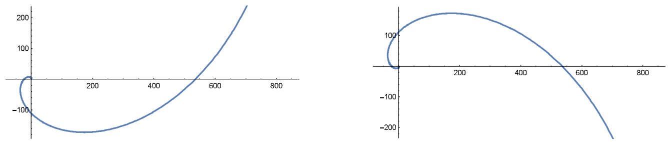


図 7. The logarithmic spiral $\gamma(t)$ and $\gamma(-t)$.

The logarithmic spiral

$$\gamma(t) := e^t(\cos t, \sin t) \qquad (t \in \mathbf{R})$$

is analytically complete. In fact a C^ω-arc proper. However, γ is not C^∞-arc proper. In fact, the smooth map

(2) $$\sigma(s) := \begin{cases} \gamma(1/s) & (\text{if } s < 0), \\ (0,0) & (\text{if } s = 0), \\ \gamma(-1/s) & (\text{if } s > 0) \end{cases}$$

connects the two sets $\gamma_1(\mathbf{R})$ and $T(\gamma_1(\mathbf{R}))$ smoothly.

Properness of S^3_1-catenoids

Proposition 13 (FKKRUYY [1]). *The catenoids of TE, TP, TH have no analytic extensions, but other five types have analytic extensions. Moreover,*

- *the catenoids of TE, TP are proper maps, and the catenoids of types TH are C^0-arc proper maps,*
- *the analytic extensions of catenoids of SH, SP, LS are proper maps, and*
- *the analytic extensions of catenoids of SE and LH are C^0-arc proper maps.*

After the analytic extensions, all of S^3_1-catenoids admit only at most countably many cone-like singular points. So they are analytically complete (by Theorem 11).

Conclusions

- Amongst three $\mathbf{R}^3_1$-catenoids, two catenoids f_P and f_H have analytic extensions, and after the analytical extensions, they are analytically complete.
- There are totally 8 types of S^3_1-catenoids. We show that analytic extension of S^3_1-catenoids except for the types TE, TH, TP.
- Moreover, we showed that after the analytic extensions, all of S^3_1-catenoids are analytically complete by showing that they are all C^ω-arc-proper.
- At the moment, we expect that
  C^ω-maps having only "generic singular points" have no "essential analytic extension" when they are C^ω-arc-proper.