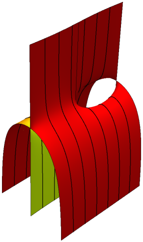
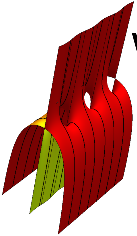
Translating solitons for Mean Curvature Flow

joint work with D. Hoffman and B. White

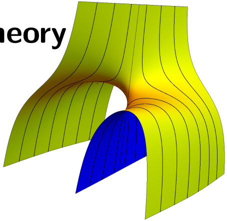
Francisco Martín, UGR





Workshop on Surface Theory
— UY60 —

March 5, 2022.



- D. Hoffman, —, B. White. Scherk-like Translators for Mean Curvature Flow. Journal of Differential Geometry, to appear.

- D. Hoffman, —, B. White. Nguyen's Tridents and the Classification of Semigraphical Translators for Mean Curvature Flow. J. Reine Angew. Math., to appear.

Translators

Definition

We say that $M \subset \mathbb{R}^3$ is a translator with **velocity** v if

$$M \mapsto M + t\, v$$

is a **mean curvature flow**.

Remark

This is equivalent to say

$$\vec{H} = v^{\perp}.$$

Up to a rigid motion and a homothety we can assume that $v = (0, 0, -1)$. Then the translator equation has the form

$$\vec{H} = (0, 0, -1)^{\perp}.$$

Translators as minimal surfaces

In 1994, T. Ilmanen observed that M is a translator iff M is minimal with respect the metric

$$g_{ij} := \mathrm{e}^{-x_3}\delta_{ij}.$$

This allows us to use:

1. compactness theorems,
2. curvature estimates,
3. maximum principles,
4. monotonicity,

for g-minimal surfaces. Moreover, reflection in vertical planes and 180^o-rotation about vertical lines are isometries of g. Therefore, we can use **Schwarz reflection** and **Alexandrov method of moving planes** in our context.

Translators as minimal surfaces

If M is a **graphical translator** and

$$\nu : M \to \mathbb{S}^2$$

its Gauss map, then

$$\langle \nu, e_3 \rangle$$

is a positive g-Jacobi field $\Longrightarrow$ M is g-STABLE.

Remark

A sequence of translating graphs will converge, subsequentially, to a translator.

Translators as minimal surfaces

Given a translator $M = \mathrm{Graph}(u)$, $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, the vertical translates of M are also g-minimal and foliate $\Omega \times \mathbb{R}$.

Proposition

M is **g-area minimizing** in $\Omega \times \mathbb{R}$, and if Ω is convex $\Longrightarrow$ M is **g-area minimizing** in $\mathbb{R}^3$.

Corollary (local area estimates)

If U is a bounded convex open subset of $\mathbb{R}^3$ disjoint from $\Gamma := \overline{M} \setminus M$, then

$$\mathrm{area}_g(M \cap U) \leq \frac{1}{2}\,\mathrm{area}_g(\partial U).$$

Translators as minimal surfaces

Theorem (curvature estimates up to the boundary)

There is a constant $C < \infty$ with the following property. Let M be translator with velocity $-s\,\mathbf{e}_3$ in $\mathbb{R}^3$ (where $s > 0$) such that

1. M is the graph of a smooth function $F : \Omega \to \mathbb{R}$ on a convex open subset Ω of $\mathbb{R}^2$.
2. $\Gamma := \overline{M} \setminus M$ is a polygonal curve (not necessarily connected) consisting of segments, rays, and lines.
3. $\overline{M}$ is a smooth manifold-with-boundary except at the corners of Γ.

If $p \in \mathbb{R}^3$, let $r(M, p)$ be the supremum of $r > 0$ such that $\mathbf{B}(p, r) \cap \partial M$ is either empty or consists of a single line segment. Then

$$|A|(M, p) \min\{s^{-1}, r(M, p)\} \leq C,$$

where $|A|(M, p)$ is the norm of the 2^{nd} f. f. of M at p.

Translators as minimal surfaces

Compactness

Let M_i, $\Gamma_i = \overline{M_i} \setminus M_i$, and Ω_i be a sequence of examples satisfying the hypotheses of the previous theorem with $s_i \equiv 1$. Suppose that the Γ_i converge (with multiplicity 1) to a polygonal curve Γ. Thus curvature estimates imply that (after passing to a subsequence) the M_i converge smoothly in $\mathbb{R}^3 \setminus \Gamma$ to a smooth translator M. By the **corollary**, M is embedded with multiplicity 1. Let M_c be a connected component of M. Note that vertical translation gives a g-Jacobi field on M that does not change sign (since M is a limit of graphs.) By the strong maximum principle, if it vanishes anywhere on M_c, it would vanish everwhere on M_c. In that case, the translator equation implies that M_c is **flat**. Thus each connected component M_c of M is either **a graph** or is **flat and vertical**.

Translating graphs

A **graphical translator** is a translator that is the graph of a function over a domain in $\mathbb{R}^2$. The **grim reaper surface**: it is the graph of the function

$$(x, y) \mapsto \log(\sin y) \tag{1}$$

over the strip $\mathbb{R} \times (0, \pi)$. Rotate the grim reaper surface about the y axis by an angle $\theta \in (-\pi/2, \pi/2)$ and then dilate by $1/\cos\theta$, the resulting surface is a also a translator. It is the graph of

$$(x, y) \mapsto \frac{\log(\sin(y\cos\theta))}{(\cos\theta)^2} + x\tan\theta. \tag{2}$$

over the strip given by $\mathbb{R} \times (0, \pi/\cos\theta)$. The graph of (2), or any surface obtained from it by translation and rotation about a vertical axis, is called a **tilted grim reaper of width** $w = \pi/\cos\theta$.

Translating graphs

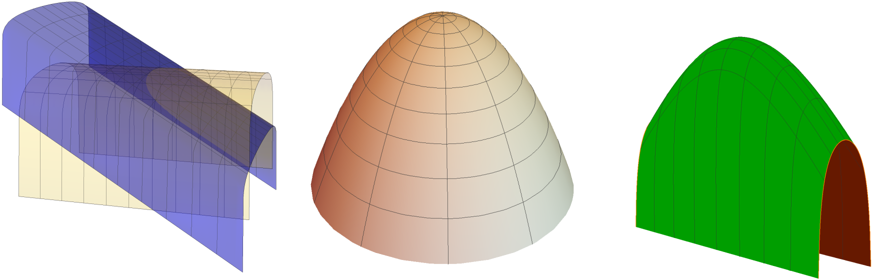


Figure: Some examples of complete graphical translators.

Theorem (Classification Theorem, Hoffman-Ilmanen-M-White)

For every $w > \pi$, there exists (up to translation) a unique complete translator $u : \mathbb{R} \times (0, w) \to \mathbb{R}$. for which the Gauss curvature is everywhere > 0. The function u is symmetric with respect to $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, w - y)$ and thus attains its maximum at $(0, w/2)$. Up to isometries of $\mathbb{R}^2$ and vertical translation, the only other complete translating graphs are the tilted grim reapers and the bowl soliton, a strictly convex, rotationally symmetric graph of an entire function.

In particular (as Spruck and Xiao had already shown), there are no complete graphical translators defined over strips of width less than π. Moreover the grim reaper surface is the only example with width π. The positively curved translator in the Classification Theorem is called a **Δ-wing**.

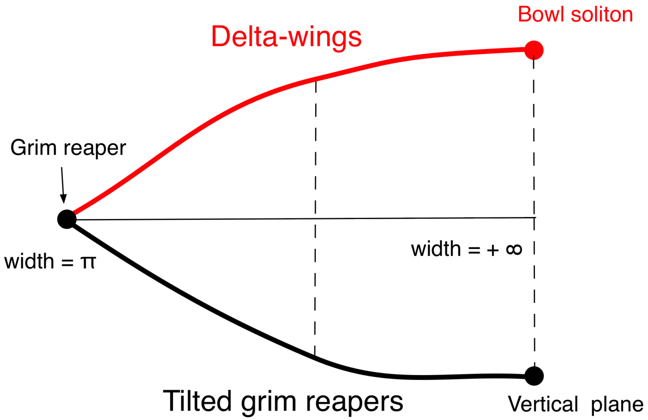


Figure: The space of complete graphical translators.

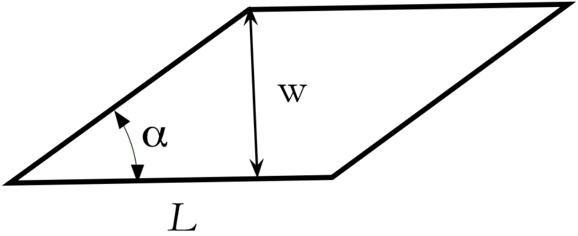
Existence Theorems

Definition

For $\alpha \in (0, \pi)$, $w \in (0, \infty)$, and $0 < L \leq \infty$, let $P(\alpha, w, L)$ be the set of points (x, y) in the strip $\mathbb{R} \times (0, w)$ such that

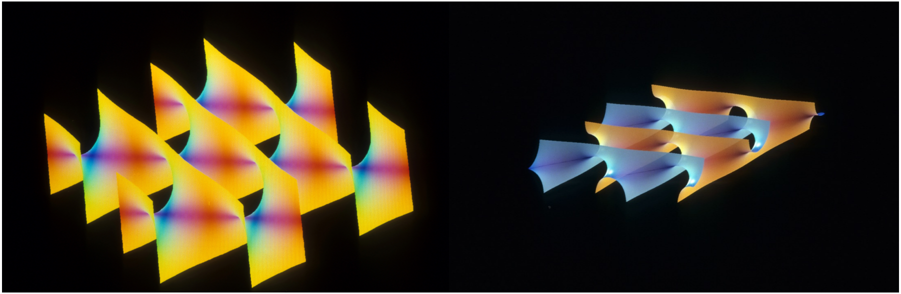
$$\frac{y}{\tan\alpha} < x < L + \frac{y}{\tan\alpha}.$$

The lower-left corner of the region is at the origin and the interior angle at that corner is α.



Classical Scherk's surfaces are obtained by solving this boundary problem:

$$(\circledast) \begin{cases} u : P = P(\alpha, w, L) \to \mathbb{R}, \\ \mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \\ u = -\infty \text{ on the horizontal sides of } P, \\ u = +\infty \text{ on the nonhorizontal sides of } P \end{cases}$$



Theorem (Classical Scherk's surfaces)

For each $\alpha \in (0, \pi)$, $w \in (0, \infty)$ and $L \in (0, \infty]$, the boundary value problem ($\divideontimes$) has a solution if and only if P is a rhombus, i.e., if and only if $L = \frac{w}{\sin \alpha}$.

- The solution is unique up to an additive constant,
- The graph of $u_{\alpha,w}$ is bounded by the four vertical lines through the corners of P.
- It extends by repeated Schwartz reflection to a doubly periodic minimal surface $\mathcal{S}_{\alpha,w}$.
- As $\alpha \to 0$, the surface $\mathcal{S}_{\alpha,w}$ converges smoothly to the parallel vertical planes $y = nw$, $n \in \mathbb{Z}$.
- As $\alpha \to \pi$, the surface $\mathcal{S}_{\alpha,w}$ converges smoothly to the helicoid given by $z = x \cot\left(\frac{\pi}{w} y\right)$.

We are interested in solving this boundary problem:

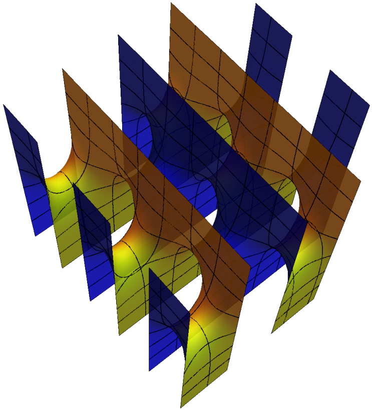
$$
(\divideontimes\divideontimes)\begin{cases}
u : P = P(\alpha, w, L) \to \mathbb{R}, \\
\mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -\frac{1}{\sqrt{1+|\nabla u|^2}}. \\
u = -\infty \text{ on the horizontal sides of } P, \\
u = +\infty \text{ on the nonhorizontal sides of } P
\end{cases}
$$

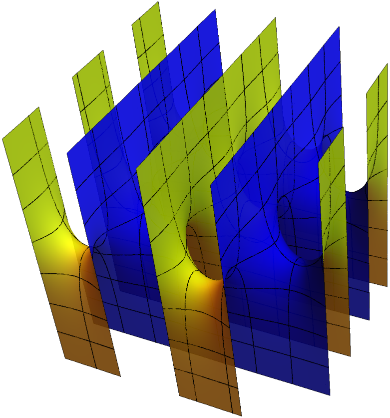
Theorem (Hoffman-M-White)

For each $\alpha \in (0, \pi)$ and $w \in (0, \infty)$, there is a unique $L = L(\alpha, w)$ in $(0, \infty]$ for which the boundary value problem (⋇⋇) has a solution.

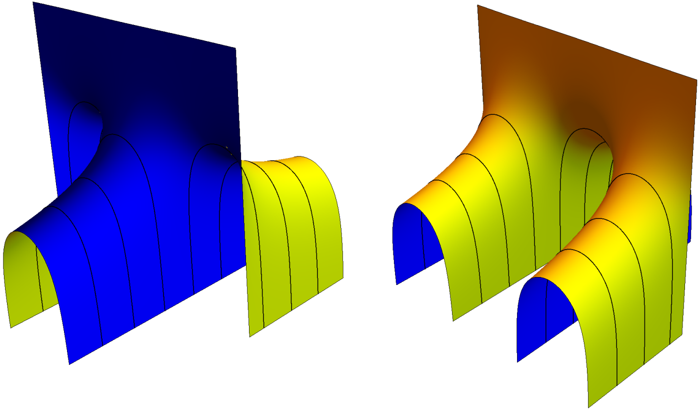
1. The length $L(\alpha, w)$ is finite if and only if $w < \pi$.
2. If $P = P(\alpha, w, L(\alpha, w))$, then the solution is unique up to an additive constant, and there is a unique solution $u_{\alpha,w}$ satisfying the additional condition
   $(\cos(\alpha/2), \sin(\alpha/2), 0)$ is tangent to the graph of u at the origin.
3. The graph of $u_{\alpha,w}$ extends by repeated Schwartz reflection to a periodic surface $\mathcal{S}_{\alpha,w}$.
   - If $w < \pi$, then $\mathcal{S}_{\alpha,w}$ is doubly periodic and we call it a **Scherk translator**.
   - If $w \geq \pi$, then $\mathcal{S}_{\alpha,w}$ is singly periodic and we call it a **Scherkenoid**.

Scherk translator $\alpha = \pi/2$, $w = \pi/2$





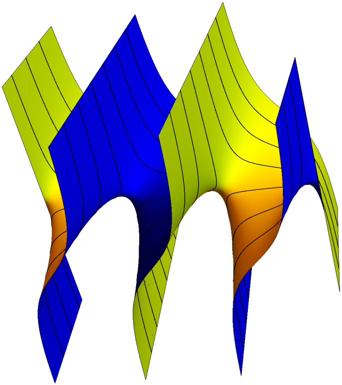
Scherkenoid $\alpha = \pi/2$, $w = \pi$

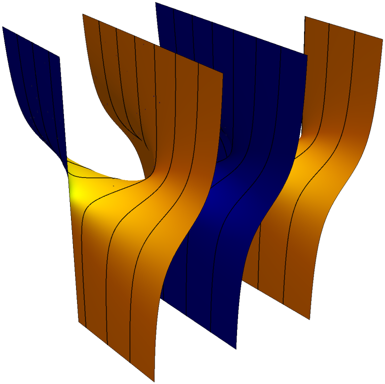


Theorem (Limit surfaces)

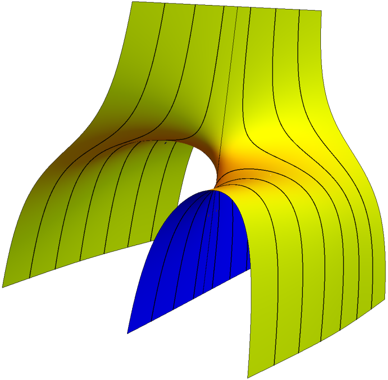
- As $\alpha \to 0$, the surface $\mathcal{S}_{\alpha,w}$ converges smoothly to the parallel vertical planes $y = nw$, $n \in \mathbb{Z}$.
- As $\alpha \to \pi$, the surface $\mathcal{S}_{\alpha,w}$ converges smoothly, perhaps after passing to a subsequence, to a limit surface M. (We do not know whether the limit depends on the choice of subsequence.) Furthermore,
  - If $w < \pi$, then M is helicoid-like: there is an $\hat{x} = \hat{x}_M \in \mathbb{R}$ such that M contains the vertical lines L_n through the points $n(\hat{x}, w)$, $n \in \mathbb{Z}$. Furthermore, $M \setminus \cup_n L_n$ projects diffeomorphically onto $\cup_{n\in\mathbb{Z}}\{nw < y < (n+1)w\}$.
  - If $w > \pi$, then M is a complete, simply connected translator such that M contains Z and such that $M \setminus Z$ projects diffeomorphically onto $\{-\pi < y < 0\} \cup \{0 < y < \pi\}$. We call such a translator a **pitchfork** of width w.
  - If $w = \pi$, then the component of M containing the origin is a pitchfork Ψ of width π, but in this case we do not know whether M is connected.

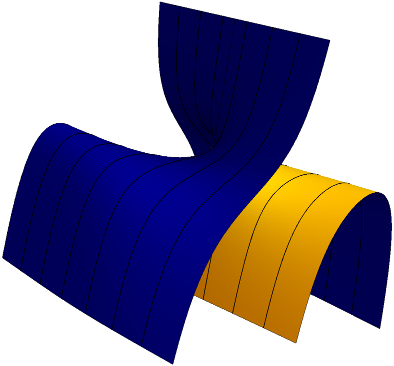
Helicoid-like translators $w = \pi/2$

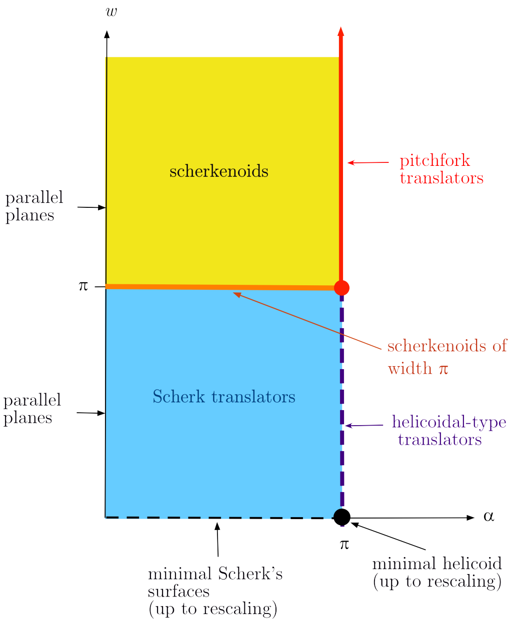




Pitchfork $w = \pi$







w
scherkenoids
parallel planes
pitchfork translators
π
scherkenoids of width π
Scherk translators
parallel planes
helicoidal-type translators
α
π
minimal Scherk's surfaces (up to rescaling)
minimal helicoid (up to rescaling)

Nguyen's Translating Tridents

Theorem (Existence)

For every $a > 0$, there is a unique translator M_a with the following properties:

1. M_a *is a smooth, properly embedded surface in* $\mathbb{R}^3$.
2. *For each integer* n, M *contains the vertical line* $\{(na, 0)\} \times \mathbb{R}$.
3. M_a *is periodic with period* $(2a, 0, 0)$.
4. $M_a \cap \{y > 0\}$ *is the graph of a function* u_a *defined on some strip* $\mathbb{R} \times (0, b)$, *with boundary values given by*

$$\begin{aligned} u_a(x,0) &= -\infty \quad \text{for } -a < x < 0, \\ u_a(x,0) &= +\infty \quad \text{for } 0 < x < a, \\ u_a(x,b) &= -\infty \quad \text{for all } x. \end{aligned}$$

5. M_a *is tangent to the* yz *plane at the origin.*

Translating tridents

Theorem (Uniqueness and limits)

If M' is any other translator with properties (1)–(4), then M' is a vertical translate of M_a.

Furthermore, the width $b = b(a)$ of the strip in (4) is a continuous, increasing function of a that takes values in $(\pi/2, \pi)$ and that tends to $\pi/2$ as $a \to 0$ and to π as $a \to \infty$. The surface M_a depends smoothly on a.

As $a \to 0$, M_a converges smoothly away from the x-axis X to the union of the xz **plane** *and the* **grim reaper surface**

$\{(x, y, z) : z = \log(\cos y)$ *and* $|y| < \pi/2\}$.

Every sequence of real numbers tending to infinity has a subsequence $a(i)$ such that $M_{a(i)}$ converges smoothly to a **pitchfork**.

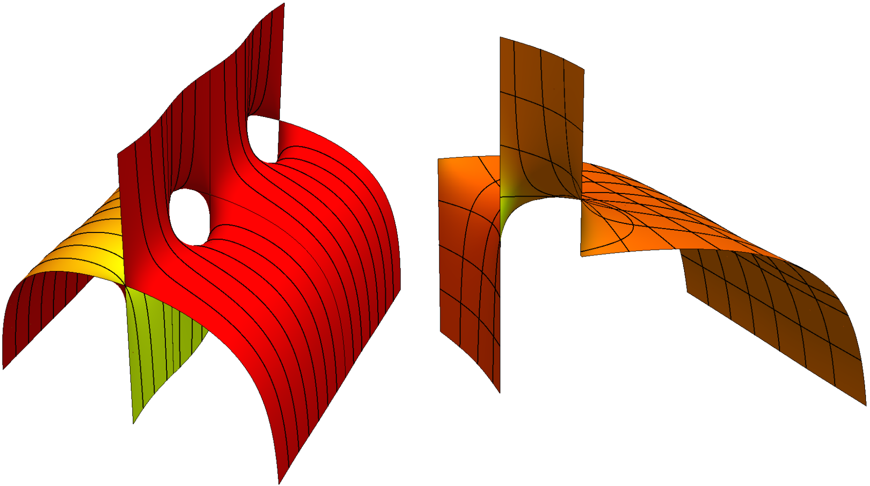


Figure: The surface M_1 .

Semigraphical Translators

Definition

A translator is M is called **semigraphical** if

(1) M is a smooth, connected, properly embedded submanifold (without boundary) in $\mathbb{R}^3$.

(2) M contains a nonempty, discrete collection of vertical lines.

(3) $M \setminus L$ is a graph, where L is the union of the vertical lines in M.

Suppose M is a **semigraphical translator**. We may suppose w.l.o.g. that M contains the z-axis Z.
Note that M is invariant under 180° rotation about each line in L, from which it follows that $L \cap \{z = 0\}$ is an additive subgroup of $\mathbb{R}^2$.
The curvature estimates imply that $M - (0, 0, \lambda)$ converges smoothly (perhaps after passing to a subsequence) to an embedded translator M_∞.

Note that the limit translator cannot have any point where the tangent plane is non-vertical. Thus M_∞ is a union of one or more parallel vertical planes.

Likewise $M_{-\infty}$ (the limit as $\lambda \to -\infty$) is the union of one or more parallel vertical planes.

Hence if Σ is a connected component of $M \setminus L$, then Σ is the graph of a function

$$u : \Omega \to \mathbb{R},$$

where Ω is one of the components of

$$\mathbb{R}^2 \setminus \Pi(M_\infty \cup M_{-\infty}).$$

Here Π is the projection $\Pi(x, y, z) = (x, y)$.

Note that such an Ω (i.e., a component of $\mathbb{R}^2$ minus two families of parallel lines) must be one of the following (after a rigid motion of $\mathbb{R}^2$):

1. A parallelogram. Such translators are called "Scherk translators" and were completely classified by Hoffman-Martín-White.
2. A semi-infinite parallelogram, i.e., a set of the form $\{(x, y) : 0 < y < w,\ x > my\}$ for some $m \neq 0$. Such translators are "Scherkenoids" and were completely classified also. In particular, for each m, there exists such a surface if and only if $w \geq \pi$, and it is unique up to vertical translation.
3. An infinite strip $\mathbb{R} \times (0, b)$ for some $b < \infty$. There are three subcases, which we discuss below.
4. A wedge, i.e., a set of the form $\{(r\cos\theta, r\sin\theta) : r > 0,\ 0 < \theta < \alpha\}$ for some α with $0 < \alpha < \pi$. This case cannot occur; we will discuss it later.
5. A halfplane. In this case, we are going to see that M contains only one vertical line. **We conjecture that this case cannot occur**.

Remark

Suppose $\Omega = \mathbb{R} \times (0, b)$ for some $0 < b \leq \infty$ (so Ω is a strip or a halfplane.) Let S be the set of points p in $\partial\Omega$ such that M contains the vertical line $\{p\} \times \mathbb{R}$. If the x-axis contains a second point $(a, 0)$ in S (in addition to the origin), then M is periodic with period $(2a, 0, 0)$ and thus the x-axis contains infinitely many points of S.

Now we discuss the case of a strip, i.e., the case when $\Omega = \mathbb{R} \times (0, b)$ for some $0 < b < \infty$.

Let S be as in the remark.. There are three subcases, according to whether S has exactly 1 point, exactly 2 points, or more than 2 points.

If $\Omega = \mathbb{R} \times (0, b)$ and S has exactly one point (namely the origin), M is a **pitchfork**.
In this case, $u(\cdot, b) = -\infty$, and u is $+\infty$ on one component of $X \setminus \{(0,0)\}$ and $-\infty$ on the other component.

A pitchfork with $\Omega = \mathbb{R} \times (0, b)$ exists if and only if $b \geq \pi$.

Conjecture

For each $b \geq \pi$, the pitchfork is unique up to rigid motions.

■ If $\Omega = \mathbb{R} \times (0, b)$ and S has exactly two points, then by the remark, one point (the origin) is on the line $y = 0$ and the other point is on the line $y = b$.

In this case, such a translator is a **helicoid-like translator**.

A helicoid-like translator with $\Omega = \mathcal{R} \times (0, b)$ exists if and only if $b < \pi$.

Conjecture

Given b, the translator is unique up to rigid motion.

Now suppose that $\Omega = \mathbb{R} \times (0, b)$ and that S contains 3 or more points.
Then S must contain more than one point on one edge of Ω, say on the edge $y = 0$. Then by the remark,

$$S \cap \{x = 0\} = \{(na, 0) : n \in \mathbb{Z}\}$$

for some $a > 0$, and M **is periodic with period** $(2a, 0, 0)$.
If S also contained a point (k, b) on the side $y = b$, then by the periodicity, it would contain $(k + na, b)$ for every n. That cannot happen.

Lemma

There is no $(2a, 0)$-periodic translator $u : \mathbb{R} \times (0, b) \to \mathbb{R}$ such that

$$u(x, 0) = u(k + x, b) = \begin{cases} -\infty & \textit{for } -a < x < 0, \textit{ and} \\ \infty & \textit{for } 0 < x < a. \end{cases}$$

Proof

Let P be a fundamental parallelogram, e.g., the parallelogram with corners $(0,0)$, $(2a,0)$, (k,b), and $(2a+k,b)$. Recall the translator equation

$$\mathrm{div}\xi = -(1+|Du|^2)^{-1/2}. \tag{3}$$

where

$$\xi = \frac{Du}{\sqrt{1+|Du|^2}}.$$

By (3) and the divergence theorem,

$$\int_{\partial P} \xi \cdot \eta \, ds < 0,$$

where η is the outward pointing unit normal.

Proof

The integrals on the left and right sides of P are equal and opposite and so cancel each other out. On the top and bottom edges of P, the integrand is 1 where $u = -\infty$ and -1 where $u = \infty$. Thus the integral is 0, **a contradiction**.
Note that because the vector field ξ is bounded, the divergence theorem holds even though there are isolated points (namely, the corners of the parallelogram) where ξ is discontinuous.

Thus if $\Omega = \mathbb{R} \times (0, b)$ is a strip and if S contains more then 2 points, then $S = \{(na, 0) : n \in \mathbb{Z}\}$ for some $a > 0$. In this case, M is the trident described before.

- When Ω is a wedge we have:

Lemma

Let $\Omega = \{(r\cos\theta, r\sin\theta) : r > 0,\ 0 < \theta < \beta\}$ where $0 < \beta < \pi$. There is no translator Σ such that

1. *Σ is a smooth, properly embedded manifold-with-boundary, the boundary being Z, and*
2. *$\Sigma \setminus Z$ is the graph a function $u : \Omega \to \mathbb{R}$.*

Proof

Suppose to the contrary that such an Σ exists. Let $W \subset \Omega \times \mathbb{R}$ be a region with piecewise smooth boundary such that $\overline{W}$ is disjoint from Z. Let W^+ and $\partial^+ W$ be the portions of W and of ∂W that lie above Σ. Let ν be the outward pointing unit normal on $\partial(W^+)$. Then

$$\begin{aligned} 0 &\geq \int_{W^+} \mathrm{div}(\mathbf{n}) \\ &= \int_{W \cap \Sigma} \mathbf{n} \cdot \nu + \int_{\partial^+ W} \mathbf{n} \cdot \nu \\ &\geq \mathrm{area}(W \cap \Sigma) - \mathrm{area}(\partial^+ W) \end{aligned}$$

since $\mathbf{n} = \nu$ on $\Sigma \cap \Omega$ and since $|\mathbf{n} \cdot \nu| \leq 1$.
It follows easily that Σ has finite entropy.

What is **entropy**?

The concept of entropy of a hypersurface in Euclidean space was introduced by Colding and Minicozzi in

Colding, Tobias H.; Minicozzi, William P., II Generic mean curvature flow I: generic singularities. Ann. of Math. (2) 175 (2012), no. 2, 755–833.

Let Σ be a surface in $\mathbb{R}^3$. Given $x_0 \in \mathbb{R}^3$ and $s_0 > 0$,we define

$$F_{x_0,s_0}[\Sigma] = \frac{1}{4\pi s_0} \int_\Sigma e^{-\frac{|x-x_0|^2}{4s_0}} \mathrm{d}\Sigma.$$

Definition

The entropy of Σ is defined as: $\lambda(\Sigma) = \sup_{x_0 \in \mathbb{R}^3, s_0 > 0} F_{x_0,s_0}[\Sigma]$

What is **entropy**?

Remark

- *(White) A surface* $\Sigma \subset \mathbb{R}^3$ *has quadratic area growth if, and only if, it has finite entropy.*
- *The entropy of a plane is* 1*. Furthermore,* $\lambda(\Sigma) \geq 1$
- *(Huisken monotonicity) If* M_t *is a mean curvature flow, then*

$$t \mapsto \lambda(M_t)$$

is a decreasing function.

Proof.

Theorem (Wedge Theorem, B. White)

Suppose W is a wedge in $\mathbb{R}^{m+1}$ with edge Γ. Suppose

$$t \in (-\infty, 0) \mapsto S(t)$$

is a self-similar, standard Brakke flow in W with boundary Γ. Then $S(\cdot)$ is a non-moving halfplane with multiplicity 1.

This implies that the tangent flow at infinity to the flow

$$t \in \mathbb{R} \mapsto \Sigma - (0, 0, t)$$

is a static, multiplicity-one halfplane. Thus by Huisken monotonicity, Σ is a flat halfplane, **a contradiction**. $\square$

- $\Omega = \mathbb{R} \times (0, +\infty)$.

Lemma

A translator $u : \{(x, y) : y > 0\} \to \mathbb{R}$ cannot be periodic in the x-direction.

Proof.

Otherwise, $(x, y) \in \mathbb{R} \times (2\pi, 3\pi) \mapsto \log(\sin y) - u(x, y)$ would attain its maximum, violating the strong maximum principle. $\square$

Classification

Theorem

If M is a semigraphical translator in $\mathbb{R}^3$, then it is one of the following:

(1) *a (doubly-periodic) Scherk translator,*

(2) *a (singly-periodic) Scherkenoid,*

(3) *a (singly-periodic) helicoid-like translator,*

(4) *a pitchfork,*

(5) *a (singly-periodic) trident, or*

(6) *(after a rigid motion) a translator containing Z such that $M \setminus Z$ is a graph over $\{(x, y) : y \neq 0\}$.*

Conjecture

Case (6) **cannot occur**.