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Time and Place: 13:20–14:20, March 6, 2022, Yokohama

Speaker: Osamu Kobayashi

Title: On an endnote of "A profile of Prof. Masaaki Umehara and Prof. Kotaro Yamada's achievements"

Abstract: The speaker would like to discuss endnote 29 of an article of Sugaku 73, MSJ, of the title.

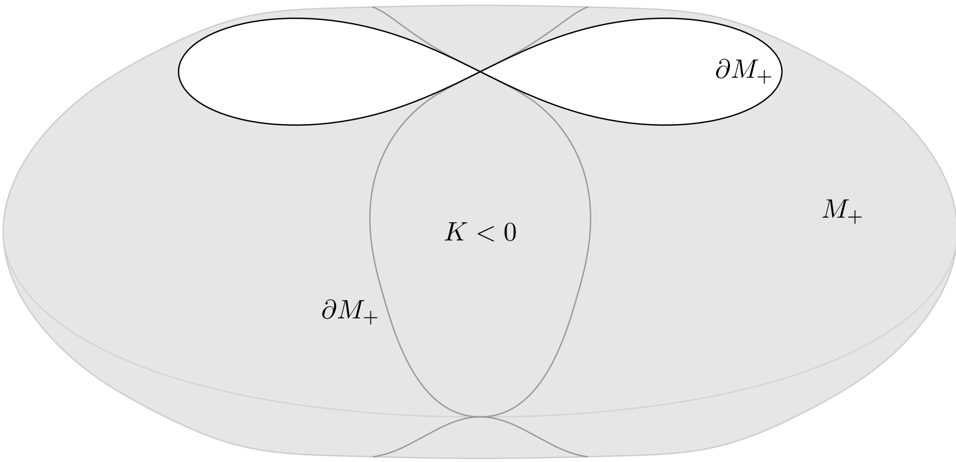

Figure 21915. $M : \left(\left(x^2+y^2\right)^2 + 4xy + 1\right)\left(4 - \sqrt{9-16z^2}\right) = 4$

Contents

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1 The Herglotz integral formula and rigidity theorems

1.1 Notation and terminology

1. **compact connected surface** (M, g)
   — M may have boundary ∂M but we often tacitly assume $\partial M = \emptyset$.

2. **the Gauss curvature** $K = K_g$

$$M_+ = \overline{\{x \in M |\ K(x) > 0\}}$$

3. **the traceless part of** $E = \{A \in T^*M \otimes T^*M |\ A(X,Y) = A(Y,X)\}$

$$\begin{aligned} \overset{\circ}{E} &= \{A \in E |\ \operatorname{tr} A = 0\} \\ A &= \frac{1}{2}(\operatorname{tr} A)g + \overset{\circ}{A} \\ \hat{A} &= \frac{1}{2}(\operatorname{tr} A)g - \overset{\circ}{A} \end{aligned}$$

4. **the cofactor tensor** $\hat{A}$ **of** A: $J \in \operatorname{End}(TM)$ is such that $J^2 = -1$ and $\operatorname{tr}_g J = 0$.

$$\begin{aligned} \hat{A}(X,Y) &= A(JX, JY) \\ \langle \hat{A}, A \rangle &= 2 \det A \end{aligned}$$

5. **Codazzi tensor** $A \in \Gamma(E)$:

$$\begin{aligned} \nabla_X A(Y,Z) &= \nabla_Z A(Y,X) \\ \operatorname{div} \hat{A} &= ({A^i}_{j;i}) = 0 \end{aligned}$$

6. **The determinant norm of** E **or polarization of** $\det$:

$$(\!(A,B)\!) := \det_g \left(\frac{A+B}{2} \right) - \det_g \left(\frac{A-B}{2} \right) = \frac{1}{2} \langle \hat{A}, B \rangle, \quad \operatorname{sgn} = (+--)$$

7. **conformal classes** $[\theta]$ **and** $[\theta]_+$**,** $\theta \in \Gamma(E)$:

$$\begin{aligned} [\theta] &= \{v\theta \in \Gamma(E) |\ v \in C^\infty(M)\} \\ [\theta]_+ &= \{e^{2u}\theta \in \Gamma(E) |\ u \in C^\infty(M)\} \subset [\theta] \end{aligned}$$

8. $\theta \in \Gamma(E)$ **is within range of** g $\overset{\text{def}}{\iff}$ $\exists \eta \in \Gamma(TM)$ such that

$$g + \frac{1}{2} L_\eta g \in [\theta]_+.$$

9. E^δ, $\delta \in C^\infty(M)$ and E_θ, $\theta \in \Gamma(E^\delta) \subset \Gamma(E)$

$E^\delta := \{\theta \in E |\ \det_g \theta = \delta\}$, $\quad E^\delta_\pm := \{\theta \in E^\delta |\ \theta$ is positive/negative definite$\}$

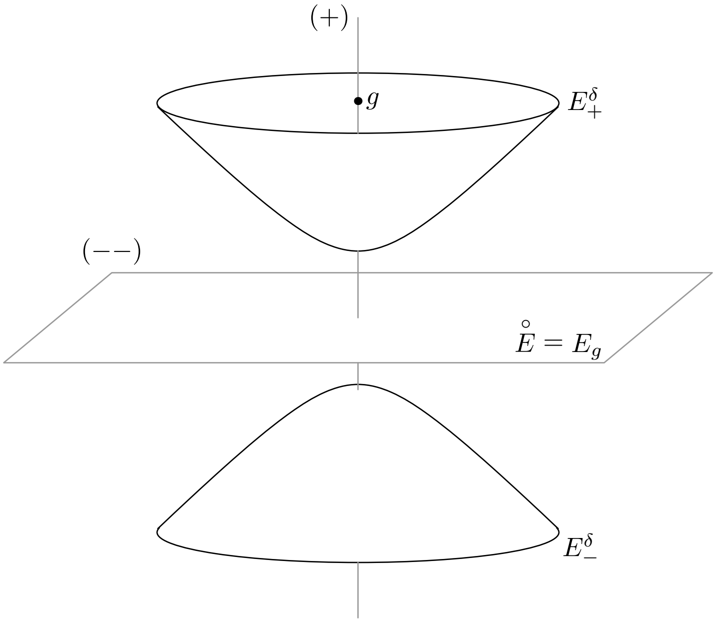

Figure 1 $\quad E \cap (T^*_p M \otimes T^*_p M)$, $\delta > 0$

$E_\theta := \{A \in E |\ ((\theta, A)) = 0\} \simeq T_\theta E^\delta$

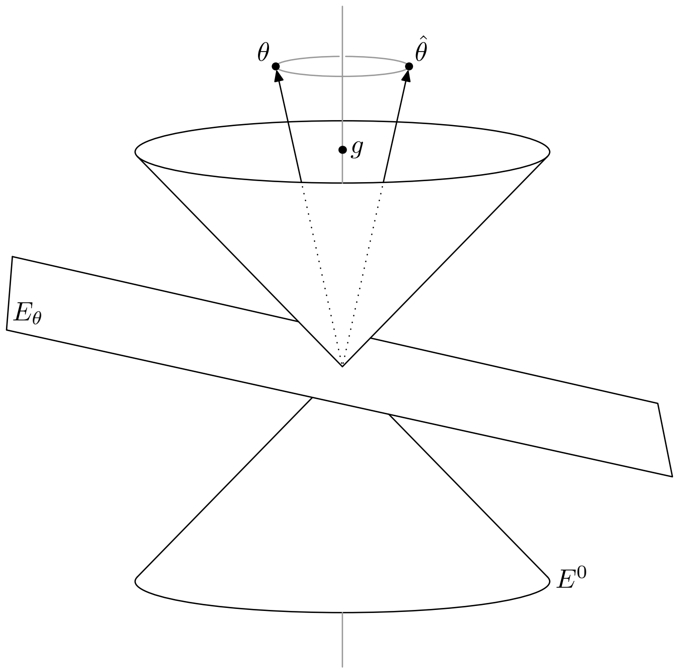

Figure 2 $\quad E_\theta \parallel T_\theta E^{\det\theta}$, $\det\theta > 0 \Rightarrow ((\ ,\ ))|E_\theta < 0$.

1.2 The Herglotz formula

Proposition 1.1 (The Herglotz integral formula)**.** *ν denotes the outward unit normal along ∂M.*

Assume : $\theta_i \in \Gamma(E^\delta)$ *are Codazzi tensors, and* $g + \frac{1}{2}L_{\eta_i} g = v_i\theta_i, \quad i = 1, 2.$

Then : $\displaystyle\int_M (v_1 + v_2) \det(\theta_1 - \theta_2)\, d\mu = \int_{\partial M} (\hat\theta_1 - \hat\theta_2)(\nu, \eta_1 - \eta_2)\, ds.$

Proof. From (1) and (2) below and the divergence theorem:

(1) $(v_1 + v_2)\det(\theta_1 - \theta_2) = ⟪\theta_1 - \theta_2, L_{\eta_1-\eta_2} g⟫$:

$$r.h.s. = ⟪\theta_1 - \theta_2, 2(v_1\theta_1 - v_2\theta_2)⟫ = ⟪\theta_1 - \theta_2, (v_1 + v_2)(\theta_1 - \theta_2)⟫ = l.h.s. \text{ because } \det\theta_1 = \det\theta_2 = \delta.$$

(2) $⟪\theta_1 - \theta_2, L_{\eta_1-\eta_2} g⟫ = \operatorname{div}((\hat\theta_1 - \hat\theta_2)(\cdot, \eta_1 - \eta_2))$:

$$⟪\theta_1 - \theta_2, L_{\eta_1-\eta_2} g⟫ = \left\langle \hat\theta_1 - \hat\theta_2, \frac{1}{2} L_{\eta_1-\eta_2} g \right\rangle = \left\langle \hat\theta_1 - \hat\theta_2, \nabla(\eta_1 - \eta_2) \right\rangle \text{ and } \operatorname{div}(\hat\theta_1 - \hat\theta_2) = 0.$$

□

Lamma 1.2. $\det\theta_1 = \det\theta_2 \Rightarrow \theta_1 - \theta_2 \in E_{\theta_1+\theta_2}$.

Proof. $⟪\theta_1 + \theta_2, \theta_1 - \theta_2⟫ = ⟪\theta_1, \theta_1⟫ - ⟪\theta_2, \theta_2⟫ = \det\theta_1 - \det\theta_2 = 0.$ □

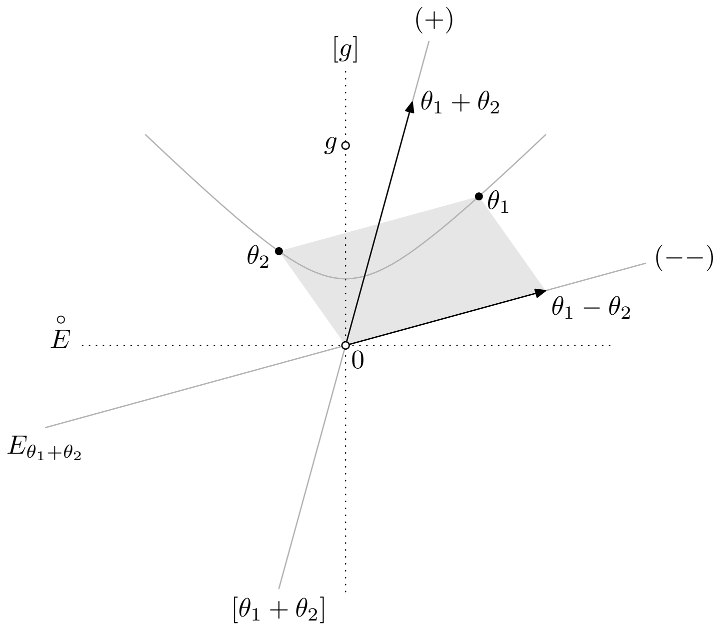

Figure 3 $\theta_1 + \theta_2 \perp_{\det} \theta_1 - \theta_2$ in E

Corollary 1.3. $\det\theta_1 = \det\theta_2$ *and* $\det(\theta_1 + \theta_2) > 0 \Rightarrow \det(\theta_1 - \theta_2) < 0$ *unless* $\theta_1 = \theta_2$.

Proof. $⟪\,,⟫$ is negative definite on $E_{\theta_1+\theta_2}$ (Figure3). □

Proposition 1.4 (The Herglotz integral formula in practicable form)**.**
Assume :

(i) $\det\theta_1 = \det\theta_2 > 0$;

(ii) *both* θ_1 *and* θ_2 *are Codazzi tensors*;

(iii) *both* θ_1 *and* θ_2 *are within range of* g;

(iv) *if* M *has boundary,*

$\det\theta_1 = \det\theta_2 = 0$ *on* ∂M ((i) *overridden) and*

∂M *consists of curves whose tangent is the nullity of both* θ_1 *and* θ_2*.*

Then : *either* $\theta_1 = \theta_2$ *or* $\theta_1 = -\theta_2$;

Proof. Replace θ_2 by $-\theta_2$ if necessary, and we may assume $\det(\theta_1 + \theta_2) > 0$.

(iv) $\Rightarrow$ the boundary integral of Proposition 1.1 $= 0$ since $\hat{\theta}_i(\nu, \cdot) = \theta_i(J\nu, J\cdot) = 0$ on ∂M.

(iii) $\Rightarrow \displaystyle\int_M (v_1 + v_2)\det(\theta_1 - \theta_2)\, d\mu = 0$ with $v_1 + v_2 > 0$.

Corollary 1.3 $\Rightarrow \theta_1 = \theta_2$. □

1.3 Rigidity theorems

Lamma 1.5 (Hadamard)**.** *If* $\partial M = \emptyset$ *and* $K > 0$*, the 2nd fundamental form of an isometric immersion of* M *in* $\mathbf{R}^3$ *is withn range of* g (Figure 4)*.*

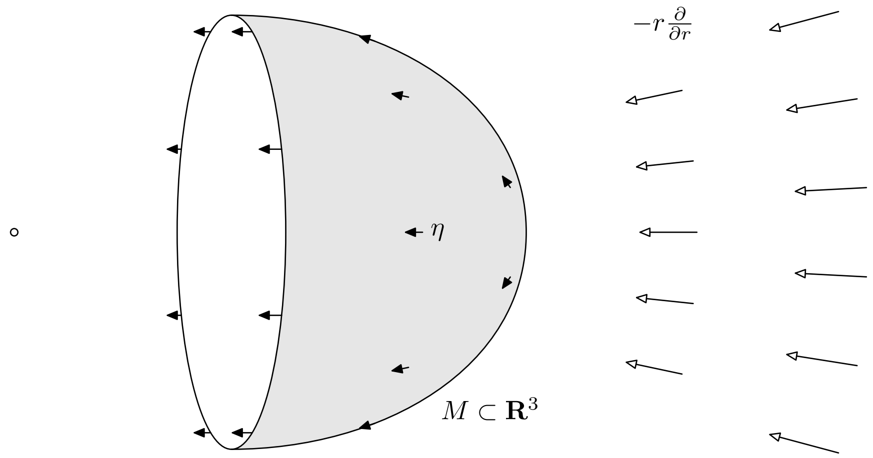

Figure 4 The condition (iii): $\eta \in \Gamma(TM)$ is the orthogonal projection of $-r\frac{\partial}{\partial r}$ to TM.

Theorem 1.6 (Cohn-Vossen 1927, Herglotz 1943)**.** *A closed surface in* $\mathbf{R}^3$ *with* $K > 0$ *is rigid.*

— pou sto: $\partial M = \emptyset$, $M \subset \mathbf{R}^3$ (immersion) $\Rightarrow M_+ \neq \emptyset$.

Lamma 1.7 (Kuiper)**.** *If $M_+ \subset \mathbf{R}^3$ is connected and $\int_{M_+} K\,d\mu = 4\pi$, the 2nd fundamental form of M_+ is within range of g and ∂M_+ consists of curvature lines of principal curvature* 0 (Figure 5)*.*

Proof. The image of the Gauss map of ∂M_+ contains only finitely many points by Kuiper's tightness argument which extends the Hadamard theorem (Lemma 1.5). $\square$

Theorem 1.8 (Alexandorov 1938)**.** *If $M_+ \subset \mathbf{R}^3$ is connected and $\int_{M_+} K\,d\mu = 4\pi$, M_+ is rigid.*

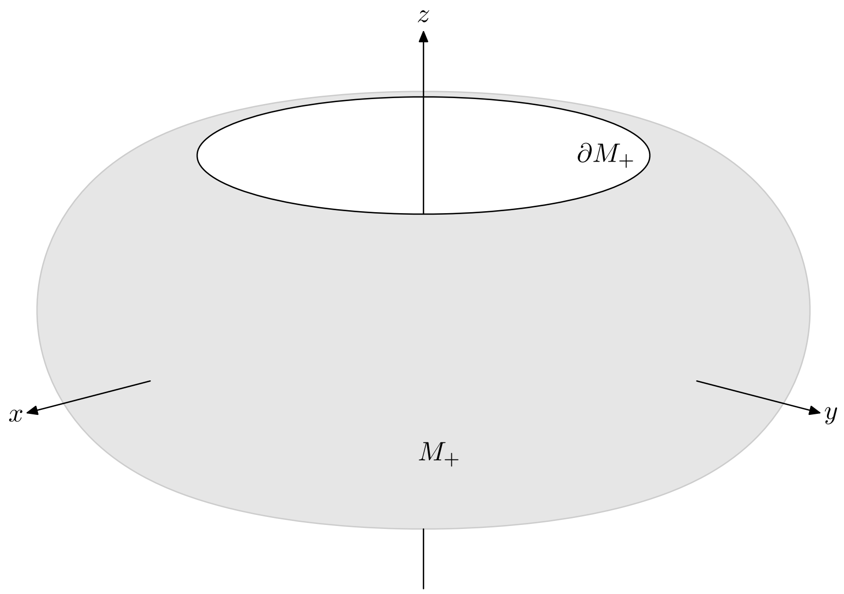

Figure 5 $\quad M_+ : \sqrt{x^2+y^2} - \sqrt{1-z^2} = \sqrt{2}$

...

Theorem 1.8 says that the maximal region M_+ of positively curved part of a surface has a tendency to be unbendable. Moreover in some examples the rigidity property of M_+ spreads over the whole of M.

Theorem 1.9 (Nirenberg 1963)**.** *A torus of revolution is rigid.*

...

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日本語訳: 小畠守男, 微分幾何学, 海外出版貿易, 1975.

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2 An intrinsic formulation

2.1 $\Sigma(M, g, \delta)$

Definition 2.1. (i) For $\delta \in C^\infty(M)$,

$$\Sigma^\delta = \Sigma(M, g, \delta) = \{\theta \in \Gamma(E^\delta) |\ \Lambda^\delta \theta = 0\},$$
$$\Lambda^\delta \colon \Gamma(E^\delta) \to \Gamma(TM);\ \Lambda^\delta \theta = \operatorname{div} \hat{\theta} = (\hat{\theta}_i{}^j{}_{;j}).$$

(ii) (the linearization of Λ^δ) For $\theta \in \Gamma(E^\delta)$,

$$\lambda_\theta \colon \Gamma(E_\theta) \to \Gamma(TM);\ \lambda_\theta \alpha = \operatorname{div} \hat{\alpha}$$

(iii) $C \colon \Gamma(E \setminus E^0) \to \Gamma(TM)$;

$$C(\theta) = \frac{\theta(\operatorname{div} \hat{\theta}, \cdot)}{\det \theta}$$

depends only on $[\theta]_+$. We say $[\theta]_+$ is *a conformally Codazzi class* if $dC(\theta) = 0$.

Lamma 2.2. (i) *$c \in \mathbf{R}$, $\theta \in \Sigma^\delta \Rightarrow c\theta \in \Sigma^{c^2\delta}$.*
(ii) *$[\theta]$ contains at most one Codazzi tensor up to a constant multiple.*
(iii) *For $\theta \in \Gamma(E \setminus E^0)$,*

$$\exists \text{ Codazzi tensor} \in [\theta]_+ \iff [\theta]_+ \text{ is a coformally Codazzi class and } [C(\theta)] = 0 \in H^1(M, \mathbf{R}).$$

(iv) *The formal adjoint of λ_θ is given as $\lambda_\theta^* \eta = -\frac{1}{2}\widehat{L_\eta g} + v\hat{\theta}$, $v = \frac{1}{|\theta|^2}\langle \theta, \nabla \eta\rangle$.*
(v) $\ker \lambda_\theta^* = \{\eta \in \Gamma(TM) |\ L_\eta g \in [\theta]\}$.

...

• Relation to the surface theory: $N(k)$ denotes 3-dim space form of constant curvature k

$$(\text{the 2nd f. f. of } (M, g) \to N(k)\,) \in \Sigma(M, g, K_g - k).$$

• Rigidity & Unbendability:

$$\theta \in \Gamma(E^\delta) \text{ is 'rigid'} \leftarrow \Sigma(M, g, \delta) = \{\pm\theta\}$$
$$\theta \in \Gamma(E^\delta) \text{ is 'unbendable'} \leftarrow \ker \lambda_\theta = 0 \text{ (near } \theta)$$

...

Proposition 2.3. *Assume $\partial M = \emptyset$.*

(i) $\chi(M) \neq 0 \Rightarrow \Sigma(M, g, -1) = \emptyset$.
(ii) $\chi(M) \neq 0 \Rightarrow \Sigma(M, g, 0) = \{0\}$.
(iii) $\chi(M) \leq 0 \Rightarrow \{\pm g\} \subset \Sigma(M, g, 1)$ *strictly, and* $\dim \ker \lambda_g = \max\{2, -3\chi(M)\}$.

Proof. (i): Poincaré-Hopf. (ii): ruled surface theorem & singular ODE.
(iii): Riemann-Roch & orientatioin coveing trick. □

The standard product metric g of T^2 is a counterexmple for the converses of (i) and (ii).

2.2 Convexity conditions

Recall:

$$[\theta]_+ = \{e^{2u}\theta \in \Gamma(E) |\ u \in C^\infty(M)\} \subset [\theta] = \{v\theta \in \Gamma(E) |\ v \in C^\infty(M)\}$$

and

$$\theta \in \Gamma(E) \textit{ is within range of } g \overset{\text{def}}{\iff} \exists \eta \in \Gamma(TM);\ g + \frac{1}{2}L_\eta g \in [\theta]_+$$
$$\iff \exists \eta \in \Gamma(TM), \exists v \in C^\infty(M),\ v > 0;\ g + \frac{1}{2}L_\eta g = v\theta.$$

Obviously

$$\theta \in [g]_+ \text{ is within range of } g.$$

This is a special case of the following:

Proposition 2.4. *The following are sufficient conditions for θ to be within range of g.*
(i) *θ is the 2nd fundamental form of $M \times \{0\}$ in $M \times \mathbf{R}$ with the metric of the form*

$$G = e^{-2t}\left(\alpha^2 dt^2 + dt \otimes \eta + \eta \otimes dt + g\right), \quad \alpha^2 = \pm v^2 + |\eta|^2.$$

θ is a Codazzi tensor if and only if $\mathrm{Ric}_G\left(\frac{\partial}{\partial t} - \eta, X\right) = 0$, $X \in TM$.
(ii) *θ is a Codazzi tensor and*

$$v_{;ij} - \frac{1}{\delta}\theta_{li;j}\hat{\theta}^{lk}v_{;k} + \left(\log\frac{\delta}{K}\right)_{;j}v_{;i} + \frac{K}{\delta}\left((\mathrm{tr}\,\theta)\,\theta_{ij} - \delta g_{ij}\right)v$$
$$= \frac{1}{\delta}\theta_{il}\left(Kg^l{}_j - K\alpha^l{}_j + \frac{K_{;j}}{K}\alpha^{lk}{}_{;k} - \alpha^{lk}{}_{;kj}\right)$$

holds for some 2-form α and $v > 0$ where $\delta = \det\theta$.
(iii) *$\partial M = \emptyset$, $\det\theta > 0$ and for some $v > 0$ the Minkowski formula*

$$\frac{1}{2}\int_M \mathrm{tr}\,\alpha\, d\mu = \int_M v(\!(\alpha, \theta)\!)\, d\mu$$

holds for all Codazzi tensor α.

...

(i): comparable to Hadamard's convexity theorem (Lemma 1.5):

$$\frac{1}{2}L_{\frac{\partial}{\partial t}}G = -G \quad \longleftrightarrow \quad \frac{1}{2}L_{-r\frac{\partial}{\partial r}}G_0 = -G_0,\ G_0 = dx^2 + dy^2 + dz^2$$

(ii): an integrability condition for the vector field η (*cf.* Lemma 2.2 (iii)).
(iii): We rely upon the theory of elliptic PDE.

...

Corollary 1.3 and thus Proposition 1.4 (i) are relevant to the ellipticity of Λ^δ or λ_θ for $\theta \in \Gamma(E^\delta)$.

2.3 Ellipticity

Proposition 2.5. (i) λ_θ *is elliptic* $\iff \det\theta > 0$.
(ii) $\partial M = \emptyset$ *and* $\det\theta > 0 \Rightarrow \dim\ker\lambda_\theta = \dim\ker\lambda_\theta^* - 3\chi(M)$.

Proof. (i): The symbol of λ_θ has non-trivial kernel $\iff \exists\xi \neq 0;\ ((\theta, \xi\otimes\xi)) = \hat{\theta}(\xi,\xi) = 0 \iff \det\theta \leq 0$.
(ii): Atiyah-Singer index formula.[*1] □

Corollary 2.6. $M = S^2$ *or* $\mathbf{R}\mathrm{P}^2 \Rightarrow \ker\lambda_g = 0$.

Proof. From Lemma 2.2 (v), $\dim\ker\lambda_g = \dim\mathrm{Conf}(M,g) = 3\chi(M)$ for $M = S^2$, $\mathbf{R}\mathrm{P}^2$. □

Standard proof uses Weizenböck formula $|\nabla\mathring{\alpha}|^2 + 2K|\mathring{\alpha}|^2 = \dfrac{1}{2}|\nabla\mathrm{tr}\,\alpha|^2 - (\hat{\alpha}^{ij}\alpha_{ij}{}^{;k})_{;k} + (\hat{\alpha}^{ij}\alpha_{ik}{}^{;k})_{;j}$ and the formulas $-\Delta\rho + K_g = e^{2\rho}\tilde{K}$ and $\mathrm{div}\,\hat{\alpha} - (\mathrm{tr}_g\alpha)d\rho = e^{2\rho}\tilde{\mathrm{div}}\,\hat{\alpha}$ for $\tilde{g} = e^{2\rho}g$.

Theorem 2.7. *If* $M = S^2$ *or* $\mathbf{R}\mathrm{P}^2$ *there is a neighborhood* $U \subset \Gamma(E_+)$ *of* $[g]_+$ *such that* $\Sigma(M, g, \det\theta) \cap U = \{\pm\theta\}$ *for* $\theta \in U$.

Proof. Apply the implicit function theorem to

$$\Psi\colon \mathrm{Diff}(M) \times C^\infty(M) \to \Gamma(E_+);\ \Psi(\phi, u) = e^{2u}\phi^* g$$

at $(\mathrm{id}, 1)$ (Figure 6) and the Herglotz formula (Proposition 1.4). □

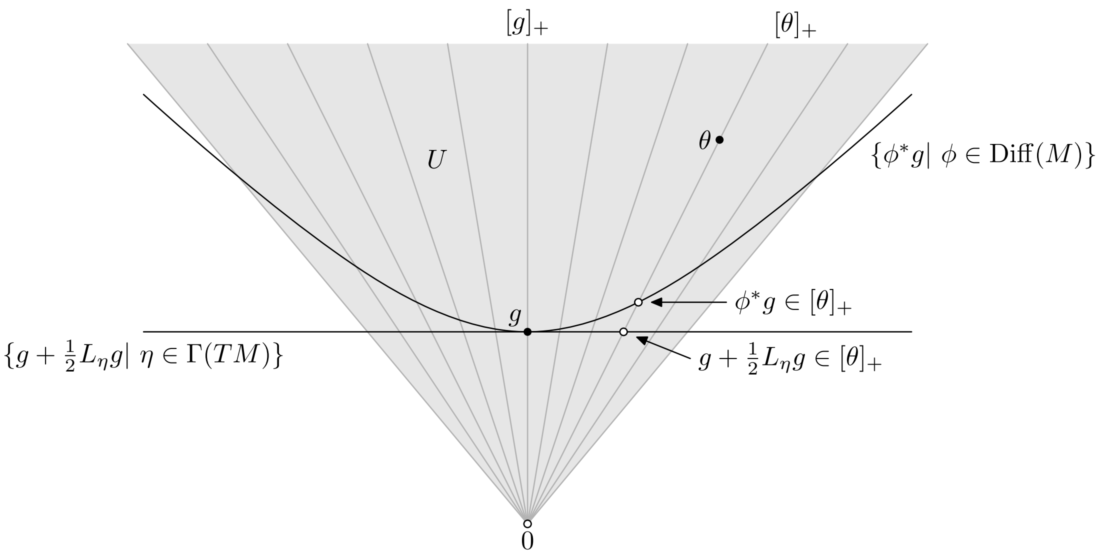

Figure 6 The orbit of $g \in \Gamma(E_+)$ by $\mathrm{Diff}(M)$ and its tangent space for $M = S^2$ or $\mathbf{R}\mathrm{P}^2$

Question 2.8. *What is the Obata equation for* $\ker\lambda_\theta^*$*?*

[*1] M. F. Atiyah and I. M. Singer, The index of elliptic operators on compact manifolds, Bull. Amer. Math. Soc. 69 (1963), 422–433.

3 The frontispiece

Figure 21915 shows the outer part of the closed surface

$$\left(\left(x^2+y^2\right)^2+4xy\right)^2+\left(\left(x^2+y^2+1\right)^2-2(x-y)^2\right)^2\left(z^2-d^2\right)=0$$

of genus 2 with $d=\frac{3}{4}$. This surface M is a union of Cassini ovals and the boundary ∂M is a pair of lemniscates of Bernoulli which are curvature lines of principal curvature zero and whose nodes are umbilic points of index -1. As is obvious from the figure and Lemma 1.5 the 2nd fundamental form is within range of the 1st fundamental form. If $\frac{3}{4}\leq d<1$, M_+ is connected. Its boundary ∂M_+ contains ∂M and other branch components from the two umbilic points which enclose negatively curved region. Accordingly the total curvature of M_+ exceeds 4π. It is interesting to know whether the surface M is bendable or not.

Proposition 2.3 (ii), Proposition 2.5 (i) and also Lemma 2.2 (ii) suggest to intuition that asymptotic curves (not drawn in the figure) in the region where $K<0$ mediate regularity property which stiffens the surface. So we ignore for a while the problem arising from negatively curved part. Then what we are interested in is how to formulate Lemma 1.7 in the style of §2. It may be appropriate to think of M_+ to be the closure of $\{x\in M|\ \delta(x)>0\}$. We see however from Lemma 2.2 (i) that the condition $\int_{M_+}\delta\,d\mu=4\pi$ does not imply that the nullity of $\theta\in\Sigma^\delta$ on ∂M_+ is tangent to ∂M_+ even if M_+ is connected, and moreover if $\theta\in\Sigma^\delta$ is within range of g. We need at least some adjustment of $\theta\in\Sigma^\delta$ by scaling. From another point of view $\theta\in\Sigma^\delta$ is a Rimannian metric of the interior $\mathring{M}_+$. In the situation of Lemma 1.7 we see that $\theta/\delta=h/K$ is a complete metric of $\mathring{M}_+$ whose area-growth is of length order if $\theta\neq 0$ on ∂M_+.

To deal with bendability question, it may be easier to consider things infinitesimally. Let $\{h_t\in\Sigma^\delta\}$ be smooth 1-parameter family and $\theta=h_0$, $\dot\theta=\frac{d}{dt}h_t|_{t=0}$, $\ddot\theta=\frac{d^2}{dt^2}h_t|_{t=0}$,..., $\theta^{(k)}=\frac{d^k}{dt^k}h_t|_{t=0}$. That is, we have the following expansion

$$h_t=\theta+t\dot\theta+\frac{t^2}{2}\ddot\theta+\frac{t^3}{3!}\theta^{(3)}+\cdots+\frac{t^k}{k!}\theta^{(k)}+\cdots$$

Then $\operatorname{div}\hat\theta^{(k)}=0$, $((\dot\theta,\theta))=0$ and $((\ddot\theta,\theta))=-\det\dot\theta$. These equalities are enough to rewrite Proposition 1.1 for infinitesimal deformations. This also makes it clearer that the Herglotz formula is a higher order variation of the Minkowski formula. Still higher variations are obtained from

$$\begin{array}{llll}
((\ddot\theta,\theta)) & =-\delta_1 & ((\theta^{(3)},\theta)) & =-\frac{3}{2}\dot\delta_1\\
((\theta^{(4)},\theta)) & =-2\ddot\delta_1+\delta_2 & ((\theta^{(5)},\theta)) & =-\frac{5}{2}\delta_1^{(3)}+\frac{5}{2}\dot\delta_2\\
((\theta^{(6)},\theta)) & =-3\delta_1^{(4)}+\frac{9}{2}\ddot\delta_2-\delta_3 & ((\theta^{(7)},\theta)) & =-\frac{7}{2}\delta_1^{(5)}+7\delta_2^{(3)}-\frac{7}{2}\dot\delta_3\\
\cdots & & &
\end{array}$$

where $\delta^{(k)}=\det\theta^{(k)}$.

Not only in some details but also on the whole we find situations analogous to zero mean curvature surfaces in Minkowski 3-space. The expansion from M_+ to whole M may be compared to a maximal surface in Minkowski 3-space which extends to a zero mean curvature surface with transition from spacelike part to timelike part.

...

4 In place of closing


poco accel.
poco rit.
Zurückhaltend
Breit.
accel.
Rubato
rit.
accel.


molto riten.
Sehr zurückhaltend
Sehr langsam u. immer noch
mehr zurückhaltend
Langsam
Molto rit.
a Tempo


Poco acceler.
83
p
87
Molto rit.
Wieder sehr langsam, wie zurvo
90
Sehr langsam
ppp
pp
96
rit.
pp subito
102


Tempo Sehr breites
Alla breve
Zurückhaltend
pp
poco cresc.
p
Etwas drängend
pp
Wieder zurückhaltend
Tempo Noch breiter als vorher
p
poco cresc.
molto cresc.
ff
Più mosso
ff
fff
poco accel.
Tempo I.
dim.
mf
molto dim.
pp

Figure 787 : From symphony no. 1 by G. Mahler